MATHEMAGIC IN THE CLASSROOM

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Contents

Introduction

Mathemagic, a branch of recreational mathematics, is a hybrid subject that combines mathematics and magic The term refers to those magic tricks that work because they are based on some aspect of mathematics. Although mathematics is rarely associated with magic and the art of conjuring, this unusual combination of mathematics and magic has an educational and entertainment value that makes mathemagic quite appropriate for use in the mathematics classroom

Historically, mathemagic can be traced at least as far back as the seventeenth century Quite a few arithmetical tricks appeared in Claude Gaspar Bachet's *Problèmes plaisons et délectables*, published in 1612 and 1624, and in Jacques Ozanam's *Créations mathématiques et physiques*, published in Paris in 1694. Some of the problems in Bachet's work have been found in the writings of earlier mathematicians such as Alcuin, Lucas Pacioli di Burgo, Tartaglia, and Cardan

Current interest in mathemagic can probably be attributed to the writings of W W. Rouse Ball and Martin Gardner Ball was a fellow in mathematics at Trinity College, Cambridge, and his classic book, *Mathematical Recreations & Essays*, which first appeared in 1892, has been a standard source of material on mathematical magic. Gardner is best known for his column "Mathematical Games," which appeared from 1957 to 1981 in *Scientific American* Many of these columns were devoted to mathemagic. His book, *Mathematics, Magic and Mystery*, first published in 1956, is probably the first attempt to survey the field of modern mathematical magic Several of his subsequent books on recreational mathematics also contain material on mathematical magic

Most of the modern contributions to mathematical magic have been made by amateur and professional magicians, not by mathematicians Mathematical tricks created by these magicians are frequently reported in books or journals devoted to conjuring and magic The mathematics on which most magic tricks are based is usually quite simple—basic concepts and principles from arithmetic, elementary algebra, and elementary number theory Consequently, tricks with a mathematical basis are of little interest to research mathematicians On the other hand, these tricks are not usually dramatic, spellbinding, or spectacular in their effects, and so they are often rejected as dull magic by many magicians

Mathematical magic nevertheless can be quite appealing in its own way. Its tricks are highly entertaining and often ingenious in design. Anyone who enjoys the challenges of mathematical puzzles and the mystique of conjuring will undoubtedly enjoy the entertainment of mathemagic.

The literature on mathematical magic is growing rapidly, and it now includes a wide variety of tricks, ranging from those that are quite complicated to perform to those which are easily mastered The magic tricks selected for this collection were chosen from the existing literature, subject to the following criteria

1. They are based entirely on mathematics
2. The mathematics on which they depend is appropriate for most middle school or secondary school students.
3. They require no feats of legerdemain or special skills possessed only by practicing magicians.
4. They can be mastered with little practice.
5. They use familiar objects such as cards, dice, coins, dollar bills, and paper clips

Many intriguing tricks have thus been excluded from this collection because they require special skills of deception or considerable practice for effective mastery

The primary reason for including magic tricks in this collection is, of course, the mathematics that makes the tricks work. Anyone having a good mastery of middle school mathematics and elementary algebra should have the mathematical background to understand the mathematical analyses of these tricks In general, most tricks of mathemagic can be explained in terms of the following mathematical ideas or processes:

1. Simple counting
2. Basic arithmetical operations
3. Operations with algebraic expressions
4. Solving algebraic equations
5. Face-value and place-value aspects of numeration
6. Nondecimal systems of numeration, especially the binary and ternary systems
7. Casting out 9's
8. Parity (the odd-even relationship between two integers)
9. Concepts from elementary number theory, such as prime factorization or divisibility

All of this mathematics either is found in the usual middle and secondary school curriculum or is easily accessible to most middle and secondary school students.

Suggestions for Using This Book

This book is written for students and teachers of middle and secondary school mathematics It is the author's hope that using mathematical magic with students will stimulate their interest in mathematics, nurture in them a positive attitude toward mathematics, and improve their skills in problem solving, critical thinking, and mathematical reasoning The author is convinced that the almost universal appeal of magic, placed in a mathematical setting, can be helpful in accomplishing these goals

Problem solving and mathematical reasoning have become increasingly more important in the mathematics curriculum for middle and secondary school students The National Council of Teachers of Mathematics in its *Curriculum and Evaluation Standards for School Mathematics* (1989) cites the ability to solve problems and the ability to reason mathematically among its five general goals for all students Investigating tricks of mathematical magic helps students to realize these two goals Students gain experience in making conjectures, collecting evidence, and forming arguments to support or reject their conjectures They look for patterns in their work, they use variables to translate their problem situations into mathematical expressions or sentences, and they apply mathematics they know to solve problems They learn to justify what they do and to communicate their results using the language of mathematics All of this occurs in the context of a situation that interests and intrigues them—mathematical magic

The tricks are classified by chapter according to the following scheme

1 Tricks of Guessing Numbers, Ages, etc.
2 Tricks with Ordinary Objects (Money, Paper Clips, Chips, etc)
3 Tricks with Dice
4 Tricks with Calendars
5 Tricks with Cards
6 Tricks of Lightning Calculations

Each trick is presented and discussed in the following format

1 Mathematical prerequisites for analyzing the trick
2. A brief description of the trick and its effect
3 Detailed directions for performing the trick
4 The secret for doing the trick

5 A mathematical analysis of why the trick works
6 Possible follow-up activities that are extensions or generalizations of the trick
7 References in the literature concerning the trick (Complete citations are in Appendix B, the bibliography.)

The Matrix of Mathematical Prerequisites for the Tricks on the following page indicates the particular concepts and skills needed for a complete analysis of why a trick works None of these concepts and skills goes beyond a standard course in first-year algebra Considered as problems to be solved, the tricks in the collection vary in their levels of difficulty. Some are easy to analyze; others are challenging In deciding whether to use a trick with a class, the teacher needs to consider the mathematical background and maturity of the students in the class so that using the trick will prove to be an appropriate and productive activity for them

The material in *Mathemagic in the Classroom* can be used in different ways, depending on the class and the teaching situation The teacher, or a student, can perform a trick for the class and then challenge the class to discover not only how the trick works but also why the trick works mathematically A mathematical analysis of why the trick works allows students to engage in problem solving that requires mathematical reasoning and critical thinking A detailed investigation of a trick often involves the use of variables, translation of verbal sentences into mathematical sentences, computations with algebraic expressions, solving equations, or the application of properties from number theory and concepts from numeration.

Many of the tricks present opportunities for follow-up activities. Variations or generalizations of the trick can be explored Students can make changes in the trick and discover what will happen because of those changes and why. Such activities give students opportunities to use their mathematical creativity They, like mathematicians doing research, are confronted with uncertainty; they do not know initially the outcome of their conjectures

If students do not have the background or skills to analyze a trick mathematically, the trick can simply be performed by the teacher or a student as an enrichment activity. This action may generate some enthusiasm for mathematics and provide practice with basic mathematical concepts or skills. Computation is frequently at the heart of performing a trick.

A single trick can be presented to a class whenever appropriate. The teacher, or a student, can present a trick as a problem-solving activity to begin a class, to end a class, or to illustrate an application of a particular mathematical concept being studied. Or, a complete "magic show" of tricks can be performed during a special class period devoted exclusively to mathemagic, with different students being responsible for different tricks. Regardless of how they are used, problem solving and mathematical reasoning, in the context of mathematical magic, should take center stage.

Matrix of Mathematical Prerequisites for the Tricks

Tricks	**ARITHMETIC**	Computation	Expanded Notation	Even and Odd Numbers	Divisibility Properties	Prime Factorization	Casting Out 9's	Nondecimal Numeration	**ALGEBRA**	Use of Variables	Simplifying Polynomials	Adding/Subtracting Polynomials	Multiplying Polynomials	Distributive Property	Factoring	Solving Equations
1 1		★								★	★			★	★	
1 2		★	★							★	★			★		
1 3		★	★				★			★	★			★	★	
1 4		★	★							★	★			★		
1 5		★	★							★	★			★		
1 6		★	★							★	★	★				
1 7		★	★							★	★	★				
1 8		★	★				★			★	★	★				
1 9		★								★	★	★		★		
2 1		★	★							★		★				★
2 2		★				★										
2 3										★						★
2 4										★		★				
2 5		★								★	★	★	★		★	
2 6		★		★												
2 7				★												
2 8		★		★												
2 9		★								★	★	★				★
2 10		★	★		★		★			★	★	★			★	
3 1		★	★							★	★			★		
3 2		★								★	★	★				
3 3		★								★	★		★	★		
3 4		★								★		★				
4 1		★								★	★	★		★		
4 2		★								★		★			★	
5 1		★	★							★	★			★		
5 2		★														
5 3		★	★							★	★			★		
5 4		★								★		★				★
5 5										★	★					
5 6										★	★	★				
5 7										★	★	★				
5 8										★	★	★				★
5 9										★	★	★				
5 10		★						★								
5 11		★						★								
6 1		★								★		★			★	
6 2		★	★							★	★					
6 3		★								★	★	★	★			
6 4		★								★	★	★		★		

CHAPTER 3

Tricks with Dice

Dice are usually associated with games—gambling games as well as board games. Dice can also be the central feature of a magic trick. Each of the magic tricks in this chapter use dice. Most of these tricks work because of the way dice are designed· On any die the sum of the two numbers on a pair of opposite faces is always seven.

3.1 Predicting Numbers on Dice

Mathematical Prerequisites

Computation, expanded notation, use of variables, simplifying polynomials, distributive property

Description

You can accurately tell a volunteer the numbers on the dice he or she has tossed after he or she gives you the final result of a series of simple calculations.

Directions for the First Trick

Turn your back to the volunteer, and give these instructions:

- Toss a pair of dice
- Multiply one of the numbers on the dice by 5
- Add 8 to the product
- Multiply the sum by 2
- Add the number on the other die to the product
- Tell me your final result

You then announce the numbers that were tossed.

Secret of the First Trick

Subtract 16 from the volunteer's final answer. The numbers on the dice are the two digits of the numeral of the resulting number

Analysis of the First Trick

Let x be the first number and y the other number. The series of calculations yields

$$(5x + 8) \cdot 2 + y$$

which simplifies to

$$10x + y + 16.$$

Subtracting 16 leaves

$$10x + y = x(10) + y(1).$$

Thus, the ten's digit and the one's digit are the numbers showing on the dice

Directions for the Second Trick

Turn your back to the volunteer, and give these instructions.

- Toss three dice.
- Multiply the number on one die by 2
- Add 5 to the product.
- Multiply the sum by 5
- Add the number on another die to the product.
- Multiply the sum by 10.
- Add the number on the third die to the product
- Tell me your final result.

You then announce the numbers that were tossed.

Secret of the Second Trick

Subtract 250 from the volunteer's final answer. The numbers on the dice are the three digits of the numeral of the resulting number.

Analysis of the Second Trick

Let x, y, and z be the numbers on the three dice The series of calculations yields

$$[(2x + 5) \quad 5 + y] \quad 10 + z$$

which simplifies to

$$100x + 10y + z + 250$$

Subtracting 250 leaves

$$100x + 10y + z = x(10^2) + y(10) + z(1).$$

Thus, the hundred's digit, ten's digit, and one's digit are the numbers showing on the three dice.

Follow-up Activity

Create different sets of instructions for tossing two dice or three dice Design a set of instructions for tossing four dice

References Ball, p 12
Gardner, *Mathematics, Magic and Mystery,* pp. 44–45

3.2 Divining the Sum of Three Dice

Mathematical Prerequisites

Computation, use of variables, simplifying polynomials, adding/subtracting polynomials

Description

You are able to tell a volunteer the sum of the numbers he or she has added from certain faces of three dice

Directions for the Trick

Turn your back to the volunteer, and give these instructions:

- Toss the three dice, and add the numbers that show on the top faces
- Choose one of the dice, and add the number on its bottom face to your sum
- Toss this die again, and add the number that turns up to your total

Turn around, and announce the volunteer's total

Secret of the Trick

When you turn around, look quickly at the numbers showing on the three dice Add those numbers, and then add 7 to that sum to get the volunteer's total

Analysis of the Trick

This trick works because on any die the sum of the two numbers on any pair of opposite faces is always 7

Let a, b, and c be the numbers that show after the first toss The volunteer's first sum is $(a + b + c)$. Suppose that the volunteer picks up the die which shows a The number on the bottom face of that die is $(7 - a)$ So, the second sum is

$$(a + b + c) + (7 - a) = b + c + 7$$

Suppose that the number d shows after that particular die is tossed again Then, the final sum is

$$(b + c + d + 7)$$

When the magician turns around, the numbers on the dice that show are b, c, and d. To get the volunteer's total, the magician only needs to add 7 to the sum $(b + c + d)$.

Follow-up Activity

Discuss how to generalize this trick for tossing n dice if $n > 3$.

References Gardner, *Mathematics, Magic and Mystery*, p. 43
Johnson and Glenn, p 267
Wyler and Ames, p 37

3.3 Prognosticating Products with Dice

Mathematical Prerequisites

Computation, use of variables, simplifying polynomials, multiplying polynomials, distributive property

Description

You can easily predict the sum of four products that a volunteer will compute from the numbers on a pair of dice

Directions for the Trick

Write 49 on a slip of paper. Fold the paper and give it to someone to hold for safekeeping Ask a volunteer to toss a pair of dice and write down the results of the following computations

- Multiply the two top numbers on the dice
- Multiply the two bottom numbers on the dice
- Multiply the top number on one die by the bottom number on the other die
- Multiply the other pair of top and bottom numbers
- Now, add up the four products, and announce the sum

Ask the person holding the folded slip of paper to unfold it and read your prediction.

Secret of the Trick

This trick is self-working The sum of the four products will always be 49. For this reason, the trick should not be repeated

Analysis of the Trick

This trick works because on any die the sum of the two numbers on any pair of opposite faces is always 7

Let a and b be the numbers that show after the dice are tossed. The four products are ab, $(7-a)(7-b)$, $a(7-b)$, and $b(7-a)$ Adding and simplifying these products yields

$$ab + (7-a)(7-b) + a(7-b) + b(7-a)$$

$$= ab + 49 - 7a - 7b + ab + 7a - ab + 7b - ab$$

$$= 49$$

Follow-up Activity

Investigate how to do this trick if three dice are tossed The volunteer will have to compute 12 products Show that the predicted sum of the 12 products must be 147.

Reference Fraser, p. 10

3.4 Prophecy with Colored Dice

Mathematical Prerequisites

Computation, use of variables, adding/subtracting polynomials

Description

You can prophesy the sum that three volunteers will generate from the numbers on three dice

Directions for the Trick

You need three dice of different colors: white, red, and green You also need three slips of paper labeled "white and red," "red and green," and "white and green," a sheet of paper, and a pencil

Write "the total is 21" on a separate slip of paper. Fold the paper, and give it to someone to hold for safekeeping. Then, ask for three volunteers, and number them Volunteer #1, Volunteer #2, and Volunteer #3

Secret of the Trick

This is another self-working trick The sum of the three numbers will always be 21 For this reason, the trick should not be repeated

Turn your back to the volunteers, and tell them to do the following

- One of you toss the three dice Then, each of you choose one of the three slips of paper
- Volunteer #1, add mentally the numbers on the two dice of the colors on your slip of paper, and write the sum on the sheet of paper.
- Volunteer #2, turn over the two dice of the colors on your slip of paper, add those numbers mentally, and write the sum on the sheet of paper
- Volunteer #3, turn over the two dice of the colors on your slip of paper, add those numbers mentally, and write the sum on the sheet of paper.
- One of you add the three numbers on the sheet of paper, and announce the sum.

Ask the person holding the folded slip of paper to unfold it and read your prediction.

Analysis of the Trick

This trick works because on any die the sum of the two numbers on any pair of opposite faces is always 7

Suppose that the colors chosen by the volunteer are

Volunteer #1	white and red
Volunteer #2	red and green
Volunteer #3	white and green ,

and suppose that after the dice are tossed the white die shows x, the red die shows y, and the green die shows z. The sums that the volunteers compute are

	White Die	**Red Die**	**Green Die**	**Volunteer's Sum**
Volunteer #1	x	y	z	$x + y$
Volunteer #2		$7 - y$	$7 - z$	$14 - y - z$
Volunteer #3	$7 - x$		z	$7 - x + z.$

The total of the three sums is

$$(x + y) + (14 - y - z) + (7 - x + z) = 21.$$

Follow-up Activity

Design a similar self-working trick that uses four volunteers and four dice of different colors. What combinations of two colors can you write on the four slips of paper so that the total is always 28?

References Fulves, *Self-Working Table Magic*, pp 31-32