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topSHELF CALCULUS

$$\frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}$$

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Contents

To the Teacher

Creative problem solving, precise reasoning, effective communication, and alertness to the reasonableness of results are some of the essential areas of mathematics that educators have specified as necessary in the development of all students to function effectively in this century. It is in the spirit of the aforementioned competencies that *Top Shelf Math* is offered to teachers of mathematics.

Top Shelf Math is intended to help students become better problem solvers. The problems presented in this format are challenging, interesting, and can easily be blended into the teaching styles and strategies of teachers who are seeking supplementary problems that support and enhance the curriculum being taught.

In general, the topics selected for each of the major content areas of *Top Shelf Math* are typical of those found in the curricula of similarly named courses offered at the high school and early college level. All of the problems presented can be used by the teacher to help his or her students improve their problem-solving skills without slowing the pace of the course in which the students are enrolled.

Other areas of utilization of the problems presented in *Top Shelf Math* could be by teachers to help prepare their students for achievement tests and advanced placement tests. Math team coaches will find these problems especially useful as they prepare their students for math competitions. It is recommended that students save the problems and solutions presented in *Top Shelf Math* because they provide a rich resource of mathematical skills and strategies that will be useful for preparing to take standardized tests and enrollment in future mathematics courses.

Mathematical thought along with the notion of problem solving is playing an increasingly important role in nearly all phases of human endeavor. The problems presented in *Top Shelf Math* help provide the teacher with a mechanism for the students to witness a variety of applications in a wide sphere of real-life settings.

In problems requiring a calculator solution, it is recommended that only College Board-approved calculators be used. In addition, some problems will suggest that a calculator not be used and that the solution will require an algebraic procedure.

The approach for solving the problems presented in *Top Shelf Math* is consistent with emphasis by national mathematics organizations for reform in mathematics teaching and learning, content, and application by taking advantage of today's technological tools that are available to most if not all high school and college students enrolled in similarly named courses. *Top Shelf Math* provides the teacher with a balance of using these tools as well as well-established approaches to problem solving.

We hope that you will find the problems useful as general information as well as in preparation for higher-level coursework and testing. For additional books in the *Top Shelf Math* series, visit our web site at walch.com.

INSTRUCTION

Derivatives and Integrals of Logarithmic Functions

Because the natural exponential function $f(x) = e^x$ is continuous and increasing on the entire real number line, it must possess an inverse function. This inverse function is called the natural logarithmic function. The domain of the natural logarithmic function is the set of positive real numbers and is defined as follows:

$$\ln x = b \text{ if and only if } e^b = x$$

Just as the natural logarithmic function was defined as the inverse of the natural exponential function, the **logarithmic function** to any positive base $a \neq 1$ is the inverse of the exponential function $f(x) = a^x$. Remember, when $a = 10$, the function given by $\log_{10}x$ is called the **common logarithmic function.**

Remember, when *a* = 10, the function given by $\log_{10}x$ is called the common logarithmic function.

Basic techniques (such as product rule, quotient rule, power rule) for finding derivatives that were provided in previous sections are repeated here for convenience. Again in these formulas, a and n are constants, and u and v are differentiable functions of x. Included here is the natural logarithmic function $f(x) = \ln x$ and the common logarithmic function $f(x) = \log_{10}x$, or more simply stated, $f(x) = \log x$.

$$\frac{d}{dx}a = 0$$

$$\frac{d}{dx}au = a\frac{du}{dx}$$

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx} \text{ (power rule)}$$

$$\frac{d}{dx}(u+v) = \frac{d}{dx}u + \frac{d}{dx}v$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \text{ (product rule)}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \text{ (quotient rule)}$$

The derivative rule for inverses: $g'(f(x)) = \frac{1}{f'(x)}$.

Methods or formulas for taking derivatives of logarithmic functions will now be expanded to the techniques listed below:

$$\frac{d}{dx}\ln u = \frac{1}{u} \cdot \frac{du}{dx},\ u > 0, \quad \frac{d}{dx}(\log_a u) = \frac{1}{(\ln a)u} \cdot \frac{du}{dx}$$

Method of Logarithmic Differentiation

1. For functions in the form of $y = u$, take the natural logarithm of both sides.
2. Use logarithmic properties to rid ln u of as many products, quotients, and exponents as possible.
3. Differentiate implicitly.
4. Solve for $\frac{dy}{dx}$.
5. Substitute for y.

As was the case for derivatives, the basic integration formulas are repeated here for convenience and extended to include integrals of the form $\log_a x$ and $\int \frac{du}{u}$. Again in these formulas, a and n are constants, and u and v are differentiable functions of x.

$$\int kf(x)dx = k\int f(x)dx, \quad \int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx, \quad \int u^n du = \frac{u^{n+1}}{n+1} + C$$

The integration formulas listed above are used in conjunction with the integration formulas for exponential functions listed below:

To evaluate integrals involving base a logarithms, we convert them to natural logarithms and use an appropriate integration technique.

$$\int \frac{du}{dx} = \ln|u| + C$$

π TRY IT Practice Activities

1. Find the exact value of $\int_1^{\sqrt{5}} \frac{\ln(x^2)}{x}\,dx$.

2. Find $\frac{dy}{dx}$ for each of the following equations:

   (a) $y = \ln[(x+1)(x+2)]$

   (b) $y = -\ln\left|\frac{1+\sqrt{1-x^2}}{x}\right|$

3. The solution of the equation $(x+1)\frac{dy}{dx} = x(y^2+1)$ is

4. A particle is moving on the curve $y = 2x - \ln 3x$ so that $\frac{dx}{dt} = -2$ at all times t. At the point (1, 2), find $\frac{dy}{dt}$.

5. Assuming y is positive, find $\frac{dx}{dt}$ by the method of logarithmic differentiation if $y = \frac{x+5}{x\cos x}$.

6. (a) $\int \frac{(x-2)^3}{x^2}dx$

(b) $\int \frac{x^3 - x - 1}{(x+1)^2}dx$

(c) $\int \frac{(1 - \ln t)^2}{t}dt$

(d) $\int \frac{2x-1}{\sqrt{4x - 4x^2}}dx$

7. Find $\frac{dy}{dx}$ of the following:

(a) $y = \ln(x\sqrt{x^2 + 1})$

(b) $y = x\ln^3 x$

(c) $x = \frac{1}{1-t}$ and $y = 1 - \ln(1-t)$, $t < 1$

8. If $f(x) = \ln x^3$, then $f''(3) =$

