topSHELF

ADVANCED ALGEBRA

$$2x+10=6x-4$$

$$\sqrt[5]{64x^{10}y^{7}}$$

$$2\div 2|x-7|=26\div 2$$

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Contents

To the Teacher

Creative problem solving, precise reasoning, effective communication, and alertness to the reasonableness of results are some of the essential areas of mathematics that educators have specified as necessary in the development of all students to function effectively in this century. It is in the spirit of the aforementioned competencies that *Top Shelf Math* is offered to teachers of mathematics.

Top Shelf Math is intended to help students become better problem solvers. The problems presented in this format are challenging, interesting, and can easily be blended into the teaching styles and strategies of teachers who are seeking supplementary problems that support and enhance the curriculum being taught.

In general, the topics selected for each of the major content areas of *Top Shelf Math* are typical of those found in the curricula of similarly named courses offered at the high school and early college level. All of the problems presented can be used by the teacher to help his or her students improve their problem-solving skills without slowing the pace of the course in which the students are enrolled.

Other areas of utilization of the problems presented in *Top Shelf Math* could be by teachers to help prepare their students for achievement tests and advanced-placement tests. Math-team coaches will find these problems especially useful as they prepare their students for math competitions. It is recommended that students save the problems and solutions presented in *Top Shelf Math* because they provide a rich resource of mathematical skills and strategies that will be useful for preparing to take standardized tests and enrollment in future mathematics courses.

Mathematical thought along with the notion of problem solving is playing an increasingly important role in nearly all phases of human endeavor. The problems presented in *Top Shelf Math* help provide the teacher with a mechanism for the students to witness a variety of applications in a wide sphere of real-life settings.

In problems requiring a calculator solution, it is recommended that only College Board-approved calculators be used. In addition, some problems will suggest that a calculator not be used and that the solution will require an algebraic procedure.

The approach for solving the problems presented in *Top Shelf Math* is consistent with emphasis by national mathematics organizations for reform in mathematics teaching and learning, content, and application by taking advantage of today's technological tools that are available to most if not all high school and college students enrolled in similarly named courses. *Top Shelf Math* provides the teacher with a balance of using these tools as well as well-established approaches to problem solving.

We hope that you will find the problems useful as general information as well as in preparation for higher-level coursework and testing. For additional books in the *Top Shelf Math* series, visit our web site at walch.com.

INSTRUCTION

Linear Equations

A linear equation normally has only one solution or value that makes the statement true.

An **equation** is a mathematical statement in which one side equals the other side. A **linear equation** is one that contains only **variables** to the first degree. That means that you will not see any variables raised to any powers other than one. A linear equation normally has only one solution or value that makes the statement true.

When solving linear equations, use inverse operations applied to BOTH sides of the equal or inequality sign. The inverse operations "undo" the operations in the problem so you can find the solution, and apply it to both sides to keep the statement balanced. Your goal when solving a linear equation is to get the variable to one side and the numbers to the other. An easy way to check to be sure that your work is correct is to plug your solution into the original equation for whatever variable you were solving for. The equation will be a true statement if your solution is correct.

Example 1

Solving a linear equation with variables on both sides.

Solve: $2x + 10 = 6x - 4$

1. In order to gather the variables on one side, subtract $2x$ from both sides. Once you do this, combine like terms:

   $2x - 2x + 10 = 6x - 2x - 4 \quad \Rightarrow \quad 10 = 4x - 4$

2. Next, to gather the numbers on the other side, add 4 to both sides and combine like terms:

   $10 + 4 = 4x \quad \Rightarrow \quad 14 = 4x$

3. Last, isolate the x completely; to get the solution, we must divide both sides by 4. Remember to reduce any fractions:

   $$\frac{14}{4} = \frac{4x}{4} \quad \Rightarrow \quad \frac{7}{2} = x$$

   The solution to the equation is $x = \frac{7}{2}$.

A few notes to remember:

- It does not matter what side the variables are on and what side the numbers are on. It is just important to separate the numbers and the variables.
- You must apply the inverse operation to every term in your equation.
- You can only combine like terms.
- Your goal is to isolate the variable, meaning get the variable by itself on one side.

Example 2

Solving a linear equation using the distributive property to get rid of parentheses.

Solve: $4(3x - 6) = -(-2x + 14)$

1. Before gathering the variable on one side, apply the distributive property to clear all parentheses:

   $12x - 24 = 2x - 14$

2. To gather the variable on one side, subtract $2x$ from both sides:

   $12x - 2x - 24 = 2x - 2x - 14 \quad \Rightarrow \quad 10x - 24 = -14$

3. Now, add 24 to both sides to move all the numbers to the right side of the equation:

   $10x - 24 + 24 = -14 + 24 \quad \Rightarrow \quad 10x = 10$

4. Last, divide both sides by 10 to isolate the variable:

   $$\frac{10x}{10} = \frac{10}{10} \quad \Rightarrow \quad x = 1$$

The solution to the equation is $x = 1$.

Remember, you can check any solution by plugging it back into the original equation. If the solution is correct, you should get a true statement when you plug the solution in for the variable. This is something that should be done for every equation you solve. Let's check your work for the preceding example:

Verify that $x = 1$ is the solution to the equation: $4(3x - 6) = -(-2x + 14)$.

1. Plug 1 in for every x in the equation:

   $4(3(1) - 6) = -(-2(1) + 14)$

2. Simplify, using the correct order of operations:

   $4(3 - 6) = -(-2 + 14) \quad \Rightarrow \quad 4(-3) = -(12)$

   $\Rightarrow -12 = -12$

3. Because $-12 = -12$, your solution is correct.

When solving equations with decimals or fractions, there are a couple of tricks you can apply to make your computation easier:

Fractions: If an equation contains one or more fractions, you can multiply every term by the least common denominator of all the fractions and clear them.

Example 3

Solving a linear equation that contains fractions.

Solve: $\frac{1}{2}x - 5 = -4x + \frac{5}{3}$

1. Examine all the denominators and determine the least common denominator:

   The least common denominator for 2 and 3 is 6.

2. Multiply every term by the least common denominator:

   $$(6)\frac{1}{2}x - (6)5 = (6)(-4x) + (6)\frac{5}{3} \quad \Rightarrow \quad 3x - 30 = -24x + 10$$

3. Finally, solve the equation following the steps from above. Be sure to check your results to be certain that your solution is correct.

Decimals: If an equation contains one or more terms with decimals, you can multiply every term by the largest power of ten required to clear the decimal from each term.

Example 4

Solving a linear equation with decimals.

Solve: $-4.2x + 12 = -6.05x + 21$

1. Examine every term that contains a decimal point and determine the smallest power of 10 that would clear every decimal point:

   Multiplying by 10 would clear the decimal in the term $-4.2x$, but would not clear the decimal point in the term $-6.05x$. Therefore, every term in this equation must be multiplied by 100 in order to clear every decimal point.

2. Multiply every term in the equation by the appropriate power of 10:

   $(100)(-4.2x) + (100)12 = (100)(-6.05x) + (100)(21) \quad \Rightarrow$

   $-420x + 1200 = -605x + 2100$

3. Finally, solve the equation following the steps from above. Be sure to check your results to be certain that your solution is correct.

Although a linear equation normally has only one solution, there are a couple of special cases to watch out for. You will know that you are dealing with one of these special cases if your variable cancels out and you end up with a statement that contains only numbers.

No solution: An equation has no solution when inverse operations cause the variable to cancel out and you are left with a false statement. This means that no value of x, or whatever variable you are using, will make the statement true. In this case, you need to indicate that the equation has no solution.

Infinite solutions: An equation has infinite solutions when inverse operations cause the variables to cancel out, leaving a true statement. This means that any value of x, or whatever variable you are using, will make the statement true. In this case, you need to indicate that the equation has infinite solutions. Sometimes an equation that has infinite solutions is referred to as an **identity**.

Examples 5 and 6

Solving equations with no solution or infinite solutions.

The following examples will illustrate the results of applying the preceding steps mentioned to equations with no solutions or infinite solutions.

Solve: $7(x + 1) - 3x = 5 + 4(x - 1)$

$7x + 7 - 3x = 5 + 4x - 4 \quad \Rightarrow \quad 4x + 7 = 1 + 4x$

$4x - 4x = 1 - 7 \quad \Rightarrow \quad 0 = -6$

The fact that the variable has canceled out indicates that this is one of the two special cases discussed above. Because 0 does not equal –6, this equation has no solution. No value of x will make the equation true.

Solve: $6x - 4 = 2(3x - 2)$

$6x - 4 = 6x - 4 \quad \Rightarrow \quad 6x - 6x = -4 + 4$

$0 = 0$

The fact that the variable has canceled out indicates that this is one of the two special cases discussed above. Because 0 does equal 0, this equation has infinite solutions. That means ANY value of x will make the equation true.

π TRY IT Practice Activities

Solve the following equations.

1. $3x - 2(4 - x) = 17$

2. $3x - \frac{1}{5} = 2x + \frac{3}{10}$

3. $6x - 4 = 2(3x + 8)$

4. $3x - 2(5x - 8) = 12x + 32$

Verify that each value is the correct solution for the given equation.

5. $\frac{2x}{3} + \frac{x}{6} = \frac{5}{2}$ $x = 3$

6. $4x - 2(5x - 8) = 5(-2x + 4)$ $x = 1$

7. Give an example of an equation that has infinite solutions.

8. Give an example of an equation that has no solution.

INSTRUCTION

Linear Inequalities

> **A linear inequality is a statement in which the two sides are not equal.**

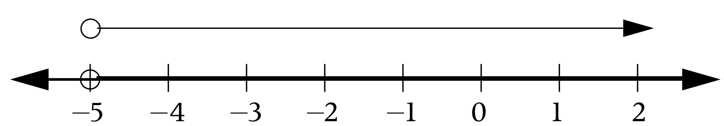
Like a linear equation, a **linear inequality** does not contain any variables raised to a power other than one. However, a linear inequality is a statement in which the two sides are not equal. One side can be less than (<), greater than (>), less than or equal to (≤), or greater than or equal to (≥) the other side. Unlike linear equations, linear inequalities often have many values of the given variables that will correctly solve the inequality. The solutions to linear inequalities normally represent a range of values that will make the given statement true.

When solving linear inequalities, use many of the same steps and procedures involved with solving linear equations. When solving a linear inequality, apply inverse operations to both sides of the inequality sign until we have isolated the variable on one side of the inequality sign.

Hint: While it does not matter what side of the inequality sign the variable is on, it is sometimes easier to comprehend the solution if the variable is isolated on the left-hand side.

The solution to a linear inequality can be represented by a graph on a number line. Closed and open circles are used in conjunction with arrows to indicate which values will solve the inequality. If the boundary number is not included in the solution, an open circle is used. If the boundary number is included in the solution, a closed circle is used. For instance:

If the solution to an inequality is $x > -5$, any value of *x greater* than 5 will solve the inequality. Since −5 is not part of the solution, draw an open circle at −5 and an arrow pointing right. All values to the right of −5 will solve the inequality.



If the solution to an inequality is $x \le 8$, any value of x less than or equal to 8 will solve the inequality. 8 is a part of the solution, so draw a closed circle at 8 and an arrow pointing left. All values to the left of 8, and including 8, will solve the inequality.

One important rule to remember when solving inequalities is:

When multiplying or dividing by a negative value, change the direction of the inequality sign.

$$-3x \leq 30 \Rightarrow \frac{-3x}{-3} \geq \frac{30}{-3}$$

Adding or subtracting values from either side of an inequality does not affect the direction of the inequality sign.

As with linear equations, the solution to a linear inequality can also be checked by plugging into the original inequality. Any value indicated by the solution should make a true statement when plugged into the inequality.

Although a linear inequality normally has only a specified range of solutions, there are a couple of special cases to watch out for. You will know that you are dealing with one of these special cases if your variable cancels out and you end up with a statement that contains only numbers.

No solution: An inequality has no solution when the inverse operations cause the variable to cancel out and you are left with a false statement. This means that no value of *x,* or whatever variable you are using, will make your statement true. In this case, you need to indicate that your inequality has no solution.

Infinite solutions: Your inequality has infinite solutions when your inverse operations cause your variables to cancel out and you are left with a true statement. This means that any value of *x,* or whatever variable you are using, will make your statement true. In this case, you need to indicate that your inequality has infinite solutions. You may also indicate that "all real numbers" will solve the inequality.

Example 1

Solve and graph the following inequality.

$3x - 22 < 5(2 - x)$

1. Apply the distributive property and then isolate the x on the left-hand side by applying inverse operations:

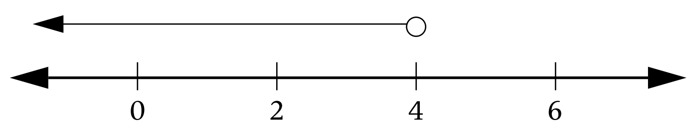
   $3x - 22 < 10 - 5x \quad \Rightarrow \quad 3x + 5x < 10 + 22$

   $8x < 32 \quad \Rightarrow \quad \frac{8x}{8} < \frac{32}{8}$

   $x < 4$

2. Determine what type of circle should be used, and graph the solution on a number line:

   Because all numbers less than 4 are only part of the solution, an open circle should be used as part of the graph. An arrow pointing left will indicate that all values to the left of 4 are part of the solution.



3. To check the solution, plug any value to the left of 4 on the number line into the original inequality:

   $2(2) - 22 < 5(2 - 2) \quad \Rightarrow \quad 4 - 22 < 5(0)$

   $-18 < 0$

   Because −18 is less than 0, our solution is correct. Any value to the left of 4, or any value less than 4, will satisfy your inequality.

Example 2

Solve and graph the following inequality.

$-18 - 5x \geq 52$

1. Isolate the x on the right-hand side by applying inverse operations to both sides of the inequality:

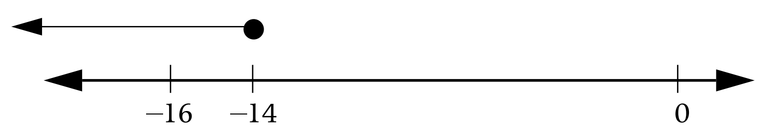
   $-18 + 18 - 5x \geq 52 + 18 \quad \Rightarrow \quad -5x \geq 70$

   Remember to change the direction of the inequality because you are dividing by a −5.

   $\frac{-5x}{-5} \leq \frac{70}{-5} \quad \Rightarrow \quad x \leq -14$

2. Determine what type of circle should be used, and graph the solution on a number line:

   Because all numbers less than −14 and including −14 are part of the solution, a closed circle should be used as part of the graph. An arrow pointing left will indicate that −14 and all values to the left of −14 are part of the solution.



Examples 3 and 4

The following examples will illustrate the results of solving a linear inequality with no solution and infinite solutions.

Example 3

Solving a linear inequality with no solutions.

Solve: $2(2x - 3) \geq 8 + 4x$

1. Distribute to clear all parentheses:

   $4x - 6 \geq 8 + 4x$

2. Apply inverse operations to both sides of the inequality sign to isolate the variable:

   $4x - 4x - 6 + 6 \geq 8 + 6 + 4x - 4x \quad \Rightarrow \quad 0 \geq 14$

   Because the variable has canceled out, you know you have a special case. Because 0 is NOT greater than 14, the statement is false, and therefore the inequality has no solutions. That means that no value of x will plug into this inequality and result in a true statement.

Example 4

Solving an inequality with all solutions.

Solve: $-3x + 14 < -3(x - 8)$

1. Distribute to clear all parentheses:

   $-3x + 14 < -3x + 24$

2. Apply inverse operations to both sides of the inequality sign to isolate the variable:

   $-3x + 3x + 14 - 14 < -3x + 3x + 24 - 14 \quad \Rightarrow \quad 0 < 10$

   Because the variable has canceled out, you know that you have a special case. Because 0 IS less than 10, the statement is true and therefore the inequality has infinite solutions. That means that ANY value of x will plug into this inequality and result in a true statement.

π TRY IT Practice Activities

Solve and graph the following inequalities.

1. $\frac{1}{2}x + 6 > 4$

2. $-7.9 < -2.1x + 4.7$

3. $4(-x + 3) \leq -5x + 5$

4. $-(5x - 7) + 10 \geq -3(x - 3)$

Determine if the following inequalities have no solution or infinite solutions.

5. $6(2x - 3) > -4(-3x + 6) - 2$

6. $\frac{1}{2}x - 16 \leq \frac{1}{4}(2x + 8)$

7. $2.4x - 3.9 < .2(12x + 7)$

8. $-2x - 6 \geq -3x - (-x + 6)$

